

EL-SHELLABILITY OF GENERALIZED NONCROSSING PARTITIONS ASSOCIATED TO WELL-GENERATED COMPLEX REFLECTION GROUPS

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ABSTRACT. In this article we prove that the poset of m -divisible noncrossing partitions is EL-shellable for every well-generated complex reflection group. This was an open problem for type $G(d, d, n)$ and for the exceptional types, for which a proof is given case-by-case.

1. INTRODUCTION

In a seminal paper [19], Germain Kreweras investigated noncrossing set partitions under refinement order. They quickly developed into a popular research topic and many interesting connections to other mathematical branches, such as algebraic combinatorics, group theory and topology, have been found. For an overview of the relation of noncrossing partitions to other branches of mathematics, see for instance [22, 27]. Many of these connections were made possible by regarding noncrossing set partitions as elements of the intersection poset of the braid arrangement. This observation eventually allowed for associating similar structures, denoted by NC_W , to every well-generated complex reflection group W . Meanwhile, these structures have been generalized even further to m -divisible noncrossing partitions, denoted by $NC_W^{(m)}$ [1, 7]. Kreweras' initial objects are obtained as the special case where W is the symmetric group and $m = 1$.

The main purpose of this paper is to prove that the poset of m -divisible noncrossing partitions possesses a certain order-theoretic property, namely EL-shellability (see Section 2.4). This is the statement of our main theorem.

Theorem 1.1. *Let $m \in \mathbb{N}$ and denote by $NC_W^{(m)}$ the poset of m -divisible noncrossing partitions associated to a well-generated complex reflection group W . Let $NC_W^{(m)} \cup \{\hat{0}\}$ be the lattice that arises from $NC_W^{(m)}$ by adding a unique smallest element $\hat{0}$. Then $NC_W^{(m)} \cup \{\hat{0}\}$ is EL-shellable for any positive integer m .*

The fact that a poset is EL-shellable implies a number of algebraic, topological and combinatorial properties. For instance, the Stanley-Reisner ring associated to an EL-shellable poset is Cohen-Macaulay. For further implications of EL-shellability we refer to [8, 10].

In the case of *real* reflection groups, Theorem 1.1 was already proved in [3] for $m = 1$ and in [1] for general m , but it has never been generalized to well-generated complex reflection groups. We recall in Section 2.1 that there are two infinite families of well-generated complex reflection groups, namely $G(d, 1, n)$ and $G(d, d, n)$, $d \geq 1$, as well as 26 exceptional groups. It follows from an observation

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of Bessis and Corran [6, p. 42] that $NC_{G(d,1,n)} \cong NC_{G(2,1,n)}$ for $d \geq 2$, and since $G(2,1,n)$ is known to be a *real* reflection group (namely the hyperoctahedral group of rank n), Theorem 1.1 follows in this case from [1, Theorem 3.7.2]. Since $G(2,2,n)$ is a *real* reflection group as well (an index 2 subgroup of $G(2,1,n)$), we only need to show Theorem 1.1 for the groups $G(d,d,n)$, $d \geq 3$, as well as for the 20 exceptional well-generated complex reflection groups that are no *real* reflection groups. In order to accomplish this, we first give an EL-labeling for NC_W where W is one of the aforementioned groups, and subsequently construct an EL-labeling for $NC_W^{(m)}$ out of it.

In Section 2 we give background information on complex reflection groups, non-crossing partitions and EL-shellability. In Section 3 we recall the concepts that Athanasiadis, Brady and Watt utilized in [3] to give an EL-labeling for the *real* reflection group $G(2,2,n)$. We generalize these concepts in Section 4 to the well-generated complex reflection groups $G(d,d,n)$, $d \geq 3$. It turns out that it is not possible to generalize the construction of [3] directly, since the crucial part in the proof of the main theorem of [3] utilizes certain properties of *real* reflection groups that do not generalize to complex reflection groups. We are still able to show the EL-shellability of $NC_{G(d,d,n)}$ by using certain properties of factorizations of the Coxeter element. These properties are elaborated in Section 5, and the proof of the EL-shellability for the case $G(d,d,n)$ is given in Section 6. For the exceptional well-generated complex reflection groups we explicitly construct an EL-labeling with the help of a computer program (see Section 7). We conclude the proof of Theorem 1.1 in Section 8 and give some applications of our main result in Section 9.

2. PRELIMINARIES

In this section we provide definitions and background for the concepts treated in this article. For a more detailed introduction to complex reflection groups, we refer to [21]. EL-shellability of partially ordered sets was introduced in [8]. More details and examples can be found there.

2.1. Complex Reflection Groups. Let V be an n -dimensional complex vector space and $w \in U(V)$ a unitary transformation on V . Define the *fixed space* $\text{Fix}(w)$ of w as the set of all vectors in V that remain invariant under the action of w . A unitary transformation is called *reflection* if it has finite order and the corresponding fixed space has codimension 1. Hence, $\text{Fix}(w)$ is a hyperplane in V , the so-called *reflecting hyperplane* of w . A finite subgroup $W \leq U(V)$ that is generated by reflections is called unitary reflection group or – as we say throughout the rest of the article – *complex reflection group*. A complex reflection group is called *irreducible* if it cannot be written as a direct product of two complex reflection groups of smaller dimensions.

According to Shephard and Todd's classification [26] of finite irreducible complex reflection groups there is one infinite family of such reflection groups, denoted by $G(d,e,n)$, with d, e, n being positive integers with $e \mid d$, as well as 34 exceptional groups, denoted by G_4, G_5, \dots, G_{37} . In case of $G(d,e,n)$, the parameter n corresponds to the dimension of the vector space V on which the group acts. We call an $(n \times n)$ -matrix that has exactly one non-zero entry in each row and column a *monomial matrix*. The group $G(d,e,n)$ can be defined as the group of monomial matrices, in which each non-zero entry is a primitive d -th root of unity and the product of all non-zero entries is a primitive $\frac{d}{e}$ -th root of unity.

For every complex reflection group W of rank n there is a set of algebraically independent polynomials $\sigma_1, \sigma_2, \dots, \sigma_n \in \mathbb{C}[X_1, X_2, \dots, X_n]$ that remain invariant under the group action. The degrees of these polynomials are called *degrees of W* . They have a close connection to the structure of W . Namely, the product of the degrees equals the group order and their sum equals the number of reflections of W plus n [21, Theorem 4.14]. We can similarly define another set of invariants, the *codegrees of W* , on the dual space V^* of linear functionals on V (see [21, Definition 10.27]). If $d_1 \leq d_2 \leq \dots \leq d_n$ denote the degrees and $d_1^* \geq d_2^* \geq \dots \geq d_n^*$ the codegrees, it follows from [23, Theorem 5.5] that a complex reflection group is *well-generated* if it satisfies

$$(1) \quad d_i + d_i^* = d_n$$

for all $1 \leq i \leq n$. We can conclude from Tables 1–4 in [14] that there are two infinite families of irreducible well-generated complex reflection groups, namely $G(d, 1, n)$ and $G(d, d, n)$, $d \geq 1$. Among the 34 exceptional complex reflection groups, 26 are well-generated. We list them in Section 7.

2.2. Regular Elements and Noncrossing Partitions. As already announced in the introduction, the objects of our concern are so-called noncrossing partitions. This section is dedicated to the definition of these objects. Let $T = \{t_1, t_2, \dots, t_N\}$ be the set of all reflections of W . Since W is generated by T , we can write every element $w \in W$ as a product of reflections. This gives rise to a length function ℓ_T that assigns to every $w \in W$ the least number of reflections that are needed to form w . More formally,

$$(2) \quad \ell_T : W \rightarrow \mathbb{N}, \quad w \mapsto \min\{k \in \mathbb{N} \mid w = t_{i_1} t_{i_2} \cdots t_{i_k}, \text{ where } 1 \leq i_1, i_2, \dots, i_k \leq N\}.$$

With the help of this length function, we can attach a poset structure to W , by defining

$$(3) \quad u \leq_T v \quad \text{if and only if} \quad \ell_T(v) = \ell_T(u) + \ell_T(u^{-1}v).$$

However, we are not interested in the complete poset (W, \leq_T) , but in certain intervals thereof. To determine these intervals, we need some more notation. Denote by V the complex vector space on which W acts. A vector $v \in V$ is called *regular* if it does not lie in one of the reflecting hyperplanes of W . If the eigenspace to an eigenvalue ζ of $w \in W$ contains a regular vector, w is called *ζ -regular*. It follows from [28, Theorem 4.2] that ζ -regular elements that have the same order are conjugate to each other. Let d_n be the largest degree of W and let ζ be a primitive d_n -th root of unity. In this case, a ζ -regular element $\gamma_\zeta \in W$ is called *Coxeter element* and by [28, Theorem 4.2(i)] has order d_n . Consider some other primitive d_n -th root of unity ξ , and some Coxeter element $\gamma_\xi \in W$ that is ξ -regular. Using a field isomorphism from $\mathbb{Q}[\zeta]$ to $\mathbb{Q}[\xi]$, we can establish a bijection between the conjugacy class of γ_ζ and the conjugacy class of γ_ξ . Hence, the Coxeter elements of a well-generated complex reflection group are conjugate up to isomorphism.

It is shown in [20] that Coxeter elements exist only in well-generated complex reflection groups. If ε denotes the identity of W and γ is a Coxeter element of W , the interval $[\varepsilon, \gamma]$ of (W, \leq_T) is called *lattice of noncrossing partitions associated to W* , and we denote it by NC_W . Since Coxeter elements are conjugate up to isomorphism and the length function ℓ_T is invariant under conjugation, the lattice structure of NC_W does not depend on a specific choice of Coxeter element. That NC_W indeed

is a lattice for every well-generated complex reflection group was shown in a series of papers [6, 11–13]. It was also shown that this lattice has a number of beautiful properties: it is for instance atomic, graded, self-dual and complemented.

In [1], Drew Armstrong introduced a more general poset that he called *poset of m -divisible noncrossing partitions* for some positive integer m . For a Coxeter element $\gamma \in W$, this poset is

$$(4) \quad NC_W^{(m)} = \left\{ (w_0; w_1, \dots, w_m) \in NC_{W^{m+1}} \mid \gamma = w_0 w_1 \cdots w_m \text{ and } \sum_{i=0}^m \ell_T(w_i) = \ell_T(\gamma) \right\},$$

where the corresponding order relation is defined as

$$(5) \quad (u_0; u_1, \dots, u_m) \leq (v_0; v_1, \dots, v_m) \text{ if and only if } u_i \geq_T v_i \text{ for all } 1 \leq i \leq m.$$

It turns out that $(NC_W^{(m)}, \leq)$ is graded with rank function $\text{rk}(w_0; w_1, \dots, w_m) = \ell_T(w_0)$ and has a unique maximal element $(\gamma; \varepsilon, \dots, \varepsilon)$. In general, however, this poset has no unique minimal element. Although Armstrong considered only *real* reflection groups¹, the same construction can be carried out in the general setting of well-generated complex reflection groups (see [7]). Not surprisingly, the case $m = 1$ yields the lattice of noncrossing partitions as defined in the previous paragraph. By theorems of several authors [4–6, 15, 16, 24], it follows that for any irreducible well-generated complex reflection group W and $m \in \mathbb{N}$ we have

$$(6) \quad |NC_W^{(m)}| = \prod_{i=1}^n \frac{md_n + d_i}{d_i},$$

where the d_i 's again denote the degrees of W in increasing order. These quantities are called *Fuß-Catalan numbers*, which we denote by $\text{Cat}^{(m)}(W)$.

Concluding this section, we give a result that we will often use in the remainder of this article. For a well-generated complex reflection group W of rank n and for some $w \in NC_W$ we have

$$(7) \quad \ell_T(w) = n - \dim \text{Fix}(w).$$

This equation can be derived from Lemma 11.30 and Proposition 11.31 in [21].

2.3. Reduced Expressions and Inversions. Let $\hat{T} \subseteq T$ be a subset of the set of all reflections of W , and let $w \in W$. We write $\ell_{\hat{T}}$ for the length function that is defined on the subgroup of W generated by \hat{T} . The sequence $(t_1, t_2, \dots, t_k) \in \hat{T}^k$ is called *shortest factorization of w* or *reduced \hat{T} -word for w* if $w = t_1 t_2 \cdots t_k$ and $\ell_{\hat{T}}(w) = k$. Moreover, if we have a partial order \preceq on \hat{T} , we say that (t_1, t_2, \dots, t_k) has a *descent at i* if $t_i \succ t_{i+1}$, for some $1 \leq i < k$. The set of all descents of (t_1, t_2, \dots, t_k) is called *descent set of (t_1, t_2, \dots, t_k)* . More generally, we say that (t_1, t_2, \dots, t_k) has an *inversion at i* if there is some $j > i$ such that $t_i \succ t_j$, and call the set of all inversions of (t_1, t_2, \dots, t_k) the *inversion set of (t_1, t_2, \dots, t_k)* .

¹A real reflection group is a reflection group that can be realized in a real vector space.

2.4. EL-Shellability of Graded Posets. Let (P, \leq) be a finite graded poset. We call (P, \leq) *bounded* if it has a unique minimal and a unique maximal element. A chain $c : x = p_0 < p_1 < \cdots < p_k = y$ in some interval $[x, y]$ of (P, \leq) is called *maximal* if there is no $q \in P$ and no $i \in \{0, 1, \dots, k-1\}$ such that $p_i < q < p_{i+1}$. Denote by $\mathcal{E}(P)$ the set of edges in the Hasse diagram of (P, \leq) . Given a poset Λ , a function $\lambda : \mathcal{E}(P) \rightarrow \Lambda$ is called *edge-labeling*. Let $\lambda(c)$ denote the sequence of edge-labels $(\lambda(p_0, p_1), \lambda(p_1, p_2), \dots, \lambda(p_{k-1}, p_k))$ of c . A maximal chain c is called *rising* if $\lambda(c)$ is a strictly increasing sequence. For some other maximal chain $c' : x = q_0 < q_1 < \cdots < q_k = y$ in the same interval, we say that c is *lexicographically smaller* than c' if $\lambda(c)$ is smaller than $\lambda(c')$ with respect to the lexicographic order on Λ^k . If λ is an edge-labeling such that for every interval of (P, \leq) there exists exactly one rising maximal chain and this chain is lexicographically smaller than any other maximal chain in this interval, we call λ an *EL-labeling*. A bounded, graded poset that admits an EL-labeling is called *EL-shellable*.

3. REVIEW OF THE CASE $G(2, 2, n)$

According to the definition of the groups $G(d, e, n)$, we see immediately that the elements of $G(2, 2, n)$ correspond to signed permutation matrices with an even number of signs. We identify signed permutation matrices with signed permutations, namely permutations π of $\{1, 2, \dots, n, -1, -2, \dots, -n\}$ that satisfy $\pi(-i) = -\pi(i)$ for every $i \in \{1, 2, \dots, n\}$. In order to write signed permutations in cycle notation, we use the abbreviations

$$\begin{aligned} ((i_1, i_2, \dots, i_k)) &:= (i_1, i_2, \dots, i_k)(-i_1, -i_2, \dots, -i_k), \\ [i_1, i_2, \dots, i_k] &:= (i_1, i_2, \dots, i_k, -i_1, -i_2, \dots, -i_k). \end{aligned}$$

Every signed permutation can uniquely be decomposed into “cycles” of the above form.

In this section we recall the usual representation of $G(2, 2, n)$ in terms of signed permutations as well as the construction of a compatible reflection ordering as defined in [3]. We generalize these constructions in Section 4 to $G(d, d, n)$ for $d \geq 3$ and apply these generalizations in the first part of the proof of Theorem 1.1 (see Section 6).

3.1. Root System. A *real* reflection group W of rank n is completely determined by a *root system*. This is a set Φ of vectors in \mathbb{R}^n which satisfies

$$(8) \quad \mathbb{R}\alpha \cap \Phi = \{-\alpha, \alpha\},$$

$$(9) \quad t_\alpha \Phi = \Phi,$$

for all $\alpha \in \Phi$, where t_α denotes the reflection in the orthogonal complement of the line $\mathbb{R}\alpha$. Fix a hyperplane \mathcal{H} in \mathbb{R}^n through the origin. We call one of the two half-spaces defined by \mathcal{H} the *positive half-space* and the other one the *negative half-space*. If \mathcal{H} does not intersect Φ , we can partition Φ into two disjoint subsets: the set of *positive roots* Φ^+ which contains all the roots in Φ that lie in the positive half-space, and the set of *negative roots* $\Phi^- = -\Phi^+$. Let $\Pi \subseteq \Phi^+$ be a vector space basis of the \mathbb{R} -span of Φ such that every $\alpha \in \Phi$ can be expressed as a linear combination of elements in Π with coefficients all of the same sign. Then Π is called *simple system* and the elements of Π *simple roots*. The existence of simple systems is for instance stated in [18, Theorem 1.3(b)].

In the present case of $G(2, 2, n)$, the usual choice for a positive system is

$$(10) \quad \Phi^+ = \{e_i \mp e_j \mid 1 \leq i < j \leq n\},$$

where e_k denotes the k -th unit vector in \mathbb{R}^n , and the usual simple system is

$$(11) \quad \Pi = \{e_i - e_{i+1} \mid 1 \leq i < n\} \cup \{e_{n-1} + e_n\}.$$

For i linearly independent roots $\alpha_1, \alpha_2, \dots, \alpha_i \in \Phi$, the intersection Φ' of Φ with the linear span of these roots is a root system itself, and is called a *rank i induced subsystem of Φ* . We can convince ourselves that $\Phi' \cap \Phi^+$ is a positive system of Φ' .

3.2. Reflections and Coxeter Element. The set of reflections of $G(2, 2, n)$ is

$$(12) \quad T = \{((i, \pm j)) \mid 1 \leq i < j \leq n\}.$$

Moreover, the set S of *simple reflections*, namely those reflections that correspond to simple roots, is

$$(13) \quad S = \{((i, i+1)) \mid 1 \leq i < n\} \cup \{((n-1, -n))\}.$$

It is well-known that a Coxeter element in a *real* reflection group is given by a product of simple reflections. Hence, if we fix the order of the simple reflections as suggested in (13) from left to right, we obtain the Coxeter element of $G(2, 2, n)$

$$(14) \quad \gamma = [1, 2, \dots, n-1][n].$$

3.3. Compatible Reflection Ordering. Let γ be a Coxeter element of W and write $NC_W(\gamma)$ if we want to emphasize the specific choice of Coxeter element γ . In order to prove the EL-shellability of the lattice of noncrossing partitions for *real* reflection groups W , Athanasiadis, Brady and Watt [3, Section 3] defined a reflection ordering for W that is compatible with γ . Subsequently, they showed that the natural labeling function

$$(15) \quad \lambda : \mathcal{E}(NC_W(\gamma)) \rightarrow T, \quad (u, v) \mapsto u^{-1}v$$

becomes an EL-labeling with respect to this ordering. We will now recall this definition. A total ordering $<$ of T is called *reflection ordering for W* if for any three distinct roots $\alpha, \alpha_1, \alpha_2 \in \Phi^+$ such that α is in the nonnegative integer span of α_1 and α_2 , the corresponding reflections satisfy either $t_{\alpha_1} < t_\alpha < t_{\alpha_2}$ or $t_{\alpha_2} < t_\alpha < t_{\alpha_1}$ (see [9, Section 5.2]). Moreover, a reflection ordering $<$ is called *compatible with a Coxeter element γ* if for any rank 2 induced subsystem $\Phi' \subseteq \Phi$, where α and β denote the simple roots of Φ' with respect to $\Phi' \cap \Phi^+$, we have that $t_\alpha t_\beta \leq_T \gamma$ implies $t_\alpha < t_\beta$ (see [3, Definition 3.1]).

According to [3, Example 3.4], the following ordering of T is a reflection ordering of $G(2, 2, n)$ that is compatible with the Coxeter element as given in (14):

$$(16) \quad \begin{aligned} ((1, 2)) &< ((1, 3)) < \dots < ((1, n-1)) < ((2, 3)) < ((2, 4)) < \dots < ((2, n-1)) \\ &< ((3, 4)) < \dots < ((n-2, n-1)) < ((1, n)) < ((1, -n)) < ((1, -2)) \\ &< ((1, -3)) < \dots < ((1, -(n-1))) < ((2, n)) < ((2, -n)) \\ &< ((2, -3)) < \dots < ((2, -(n-1))) < ((3, n)) < ((3, -n)) \\ &< ((3, -4)) < \dots < ((n-1, n)) < ((n-1, -n)). \end{aligned}$$

4. GENERALIZATION TO THE CASE $G(d, d, n)$, $d \geq 3$

In the previous section we have recalled the necessary groundwork that allows us to give an analogous construction of a compatible reflection ordering for the complex reflection groups $G(d, d, n)$. In order to do so, we first elaborate a representation of the group elements as colored permutations and then generalize the notion of a compatible reflection ordering as recalled in Section 3.3. Note that all the results and constructions given in this section agree in the case $d = 2$ with the analogous results of the previous section.

4.1. A Group of d -Colored Permutations. Remember that the elements of $G(d, d, n)$ are monomial matrices whose non-zero entries are primitive d -th roots of unity and the product of all non-zero elements is 1. In this section, we will explain how these groups can be represented as certain subgroups of S_{dn} . We will accompany this construction with the running example of $G(3, 3, 3)$.

Consider the set

$$(17) \quad \{1^{(0)}, 2^{(0)}, \dots, n^{(0)}, 1^{(1)}, 2^{(1)}, \dots, n^{(1)}, \dots, 1^{(d-1)}, 2^{(d-1)}, \dots, n^{(d-1)}\}$$

of integers with d colors. For all integers $1 \leq i \leq n$ and $0 \leq s < d$, identify the colored integer $i^{(s)}$ with the vector $(0, 0, \dots, \zeta_d^s, \dots, 0)^T \in \mathbb{C}^n$, where $\zeta_d = e^{2\pi\sqrt{-1}/d}$ is a primitive d -th root of unity and the non-zero entry appears in the i -th position. Hence, $G(d, d, n)$ is isomorphic to a subgroup of the group of permutations of the set (17) that consists of elements w that satisfy

$$w(i^{(s)}) = \pi(i)^{(s+t_i)},$$

for some $\pi \in S_n$ and $t_i \in \mathbb{Z}$ which depend on w , and the addition in the superscript is understood modulo d . Moreover, the t_i 's have to satisfy the property

$$\sum_{i=1}^n t_i \equiv 0 \pmod{d}.$$

This allows us to represent the elements of $G(d, d, n)$ in a permutation-like fashion as

$$\begin{pmatrix} 1^{(0)} & 2^{(0)} & \dots & n^{(0)} \\ \pi(1)^{(t_1)} & \pi(2)^{(t_2)} & \dots & \pi(n)^{(t_n)} \end{pmatrix}.$$

Analogously to the classical case of signed permutations, we introduce the abbreviations

$$\begin{aligned} ((i_1^{(t_1)} \dots i_k^{(t_k)})) &:= (i_1^{(t_1)} \dots i_k^{(t_k)})(i_1^{(t_1+1)} \dots i_k^{(t_k+1)}) \dots (i_1^{(t_1+d-1)} \dots i_k^{(t_k+d-1)}), \\ [i_1^{(t_1)} \dots i_k^{(t_k)}]_s &:= (i_1^{(t_1)} \dots i_k^{(t_k)} i_1^{(t_1+s)} \dots i_k^{(t_k+s)} \dots i_1^{(t_1+(d-1)s)} \dots i_k^{(t_k+(d-1)s)}). \end{aligned}$$

We can convince ourselves that every element of $G(d, d, n)$ can uniquely be decomposed into “cycles” of the above form. For a better readability, we write $[i_1^{(t_1)} \dots i_k^{(t_k)}]$ instead of $[i_1^{(t_1)} \dots i_k^{(t_k)}]_1$.

Example 4.1. Consider the group $G(3, 3, 3)$ and the element

$$w = \begin{pmatrix} 1^{(0)} & 2^{(0)} & 3^{(0)} \\ 2^{(1)} & 1^{(2)} & 3^{(0)} \end{pmatrix}.$$

The corresponding matrix representation is

$$\varphi(w) = \begin{pmatrix} 0 & \zeta_3^2 & 0 \\ \zeta_3 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where φ denotes the isomorphism that maps colored permutations to elements of $G(d, d, n)$. Hence, w acts as follows on $\{1^{(0)}, 2^{(0)}, 3^{(0)}, 1^{(1)}, 2^{(1)}, 3^{(1)}, 1^{(2)}, 2^{(2)}, 3^{(2)}\}$:

$$\begin{aligned} w(1^{(0)}) &= 2^{(1)}, & w(2^{(0)}) &= 1^{(2)}, & w(3^{(0)}) &= 3^{(0)}, \\ w(1^{(1)}) &= 2^{(2)}, & w(2^{(1)}) &= 1^{(0)}, & w(3^{(1)}) &= 3^{(1)}, \\ w(1^{(2)}) &= 2^{(0)}, & w(2^{(2)}) &= 1^{(1)}, & w(3^{(2)}) &= 3^{(2)}. \end{aligned}$$

The cycle decomposition of w is given by $((1^{(0)}2^{(1)}))$.

4.2. Reflections and Coxeter Element. The reflections in $G(d, d, n)$ are those unitary transformations that have a fixed space of codimension 1. Hence they are monomial matrices that have $n - 1$ eigenvalues equal to 1 and one eigenvalue equal to a primitive d -th root of unity that is not 1. We can conclude that $n - 2$ diagonal entries of such a matrix must be 1 and the other two diagonal entries are zero. Say, the i -th and j -th diagonal entry of a given reflection $t \in G(d, d, n)$ are equal to zero. Since t is a monomial matrix, the non-zero entry in row i is in column j and vice versa for row j . Thus, t exchanges the i -th and j -th entry of a vector $v \in \mathbb{C}^n$ and multiplies these entries with the respective roots of unity that appear at position (j, i) or (i, j) in t respectively. Hence, the reflections of $G(d, d, n)$ correspond to colored transpositions of the form $((i^{(0)}j^{(s)}))$, where $1 \leq i < j \leq n$ and $0 \leq s < d$. Clearly, there are $d \cdot \binom{n}{2}$ reflections in $G(d, d, n)$.

In analogy to (13), we will emphasize a certain subset of the set T of all reflections, namely the reflections

$$(18) \quad ((1^{(0)}2^{(0)})), ((2^{(0)}3^{(0)})), \dots, ((n-1)^{(0)}n^{(0)}), ((n-1)^{(0)}n^{(1)}),$$

call them *simple reflections*, and denote them by s_1, s_2, \dots, s_n where we fix their order as given above. The product $\gamma = s_1 s_2 \dots s_n$ is the group element

$$(19) \quad \gamma = [1^{(0)}2^{(0)} \dots (n-1)^{(0)}][n^{(0)}]^{-1},$$

for which we can show that it is a Coxeter element of $G(d, d, n)$. This will be the choice of Coxeter element to which we refer throughout the rest of the paper.

Example 4.2. The Coxeter element of $G(3, 3, 3)$ according to (19) is

$$\varphi(\gamma) = \begin{pmatrix} 0 & \zeta_3 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \zeta_3^2 \end{pmatrix}.$$

4.3. Compatible Reflection Ordering. Before we proceed to define a compatible reflection ordering for $G(d, d, n)$, we make an observation. While in the case of *real* reflection groups, every $t \in T$ satisfies $t \leq_T \gamma$ for a Coxeter element γ , this is in general not true for complex reflection groups. Consider for instance the group $G(3, 3, 3)$. Equation (6) implies that $NC_{G(3,3,3)}$ has 18 elements. Since this lattice is graded of rank 3 and self-dual, only 8 of the 9 reflections of $G(3, 3, 3)$ are contained in this lattice. Thus, we need to characterize the reflections that are contained in $NC_{G(d,d,n)}$.

Proposition 4.1. *Let γ be the Coxeter element of $G(d, d, n)$ as given in (19). For a reflection $t = ((i^{(0)}j^{(s)}))$ of $G(d, d, n)$ we have*

$$t \not\leq_T \gamma \quad \text{if and only if} \quad j < n \text{ and } 1 \leq s < d - 1.$$

Proof. Let us recall from (7) that $\ell_T(w) = n - \dim \text{Fix}(w)$, where $w \in W$ and n denotes the rank of W . We can also check that the reflections in $G(d, d, n)$ are involutions. Thus it remains to determine, for which $t \in T$ we have $\dim \text{Fix}(t\gamma) = 1$. Given an arbitrary vector $x = (x_1, x_2, \dots, x_n)^\top \in \mathbb{C}^n$, the action of γ is given by

$$(20) \quad \gamma x = (\zeta_d x_{n-1}, x_1, x_2, \dots, x_{n-2}, \zeta_d^{d-1} x_n)^\top.$$

In the following, we will investigate the action of the reflections on (20). We distinguish three cases:

(i) $t = ((1^{(0)}n^{(s)}))$, where $0 \leq s < d$. We have

$$t(\gamma x) = (\zeta_d^{d-s-1} x_n, x_1, \dots, x_{n-2}, \zeta_d^{s+1} x_{n-1})^\top.$$

Hence, $\dim \text{Fix}(t\gamma) = 1$, which implies that $t \leq_T \gamma$.

(ii) $t = ((i^{(0)}n^{(s)}))$, where $1 < i < n$ and $0 \leq s < d$. In this case, we obtain

$$t(\gamma x) = (\zeta_d x_{n-1}, x_1, \dots, x_{i-2}, \zeta_d^{d-s-1} x_n, x_i, \dots, x_{n-2}, \zeta_d^s x_{i-1})^\top.$$

Hence, again, $\dim \text{Fix}(t\gamma) = 1$, and thus $t \leq_T \gamma$.

(iii) $t = ((i^{(0)}j^{(s)}))$, where $1 \leq i < j < n$ and $0 \leq s < d$. The action of t in this case is given by

$$t(\gamma x) = (\zeta_d x_{n-1}, x_1, \dots, x_{i-2}, \zeta_d^{d-s} x_{j-1}, x_i, \dots, x_{j-2}, \zeta_d^s x_{i-1}, x_j, \dots, x_{n-2}, \zeta_d^{d-1} x_n)^\top$$

such that we obtain the equations

$$\begin{aligned} \zeta_d^{s+1} x_{n-1} &= \zeta_d^s x_1 = \dots = \zeta_d^s x_{i-1} = x_j = \dots = x_{n-1}, \\ \zeta_d^{d-s} x_{j-1} &= x_i = \dots = x_{j-1}, \\ \zeta_d^{d-1} x_n &= x_n. \end{aligned}$$

The first line has a nontrivial solution only if $s = d-1$ (which forces the components in lines 2 and 3 to be zero). This means that $\dim \text{Fix}(t\gamma) = 1$. Similarly, the same holds for the second line if $s = 0$.

For other values of s all components have to be zero, which concludes the proof. \square

For a Coxeter element γ of $G(d, d, n)$, denote by T_γ the set of all reflections $t \in T$ that satisfy $t \leq_T \gamma$.

Let $t_1, t_2 \in T_\gamma$ be non-commuting reflections and denote by $I(t_1, t_2)$ the interval of smallest rank in $NC_{G(d, d, n)}$ that contains t_1 and t_2 . If $I(t_1, t_2)$ has rank 2, we know that either $t_1 t_2 \leq_T \gamma$ or $t_2 t_1 \leq_T \gamma$.

Definition 4.1. *Let γ be a Coxeter element of $G(d, d, n)$. We call an ordering \prec of T_γ a γ -compatible reflection ordering if for all non-commuting reflections $t_1, t_2 \in T_\gamma$ such that $I(t_1, t_2)$ has rank 2, there are exactly two reflections $\tilde{t}_1, \tilde{t}_2 \in T_\gamma \cap I(t_1, t_2)$ such that $\tilde{t}_1 \tilde{t}_2 \leq_T \gamma$ implies $\tilde{t}_1 \prec \tilde{t}_2$.*

Note that this definition is a generalization of [3, Definition 3.1] as recalled in Section 3.3. However, in contrast to [3, Definition 3.1], we do not use root systems. Root systems are defined for complex reflection groups as well (see for instance

[21, Definition 1.43]), but in the present case they turn out not to be such a powerful tool as they are in the case of *real* reflection groups.

Lemma 4.1. *Let γ be the Coxeter element as defined in (19). The following ordering of T_γ is a γ -compatible reflection ordering for $G(d, d, n)$.*

$$\begin{aligned}
 (21) \quad & ((1^{(0)}2^{(0)})) < ((1^{(0)}3^{(0)})) < \dots < ((1^{(0)}(n-1)^{(0)})) \\
 & < ((2^{(0)}3^{(0)})) < \dots < ((2^{(0)}(n-1)^{(0)})) \\
 & < ((3^{(0)}4^{(0)})) < \dots < (((n-2)^{(0)}(n-1)^{(0)})) \\
 & < ((1^{(0)}n^{(0)})) < ((1^{(0)}n^{(d-1)})) < \dots < ((1^{(0)}n^{(1)})) \\
 & < ((1^{(0)}2^{(d-1)})) < \dots < ((1^{(0)}(n-1)^{(d-1)})) \\
 & < ((2^{(0)}n^{(0)})) < ((2^{(0)}n^{(d-1)})) < \dots < ((2^{(0)}n^{(1)})) \\
 & < ((2^{(0)}3^{(d-1)})) < \dots < ((2^{(0)}(n-1)^{(d-1)})) \\
 & < ((3^{(0)}n^{(0)})) < ((3^{(0)}n^{(d-1)})) < \dots < ((n-1)^{(0)}n^{(1)}).
 \end{aligned}$$

Proof. We need to show that the given ordering satisfies the properties given in Definition 4.1. In order to do that, we explicitly write down the sets $I(t_1, t_2) \cap T_\gamma$ for all non-commuting reflections $t_1, t_2 \in T_\gamma$ such that $I(t_1, t_2)$ has rank 2. These are the following six cases, where the parameters satisfy $1 \leq i < j < k < n$ and $0 \leq s < d$.

- (i) $\{((i^{(0)}j^{(0)})), ((j^{(0)}k^{(0)})), ((i^{(0)}k^{(0)}))\}$
- (ii) $\{((j^{(0)}k^{(0)})), ((i^{(0)}j^{(d-1)})), ((i^{(0)}k^{(d-1)}))\}$
- (iii) $\{((i^{(0)}j^{(0)})), ((i^{(0)}k^{(d-1)})), ((j^{(0)}k^{(d-1)}))\}$
- (iv) $\{((i^{(0)}j^{(0)})), ((i^{(0)}n^{(s)})), ((j^{(0)}n^{(s)}))\}$
- (v) $\{((i^{(0)}j^{(d-1)})), ((i^{(0)}n^{(s)})), ((j^{(0)}n^{(s+1)}))\}$
- (vi) $\{((i^{(0)}n^{(0)})), ((i^{(0)}n^{(1)})), \dots, ((i^{(0)}n^{(d-1)}))\}$

We prove the existence of two unique reflections \tilde{t}_1, \tilde{t}_2 that satisfy $\tilde{t}_1\tilde{t}_2 \leq_T \gamma$ and $\tilde{t}_1 < \tilde{t}_2$ for case (iii). The remaining cases can be shown analogously. We can verify the following analogously to the proof of Proposition 4.1:

$$\begin{aligned}
 & ((i^{(0)}j^{(0)}))((j^{(0)}k^{(d-1)})) \leq_T \gamma, \\
 & ((j^{(0)}k^{(d-1)}))((i^{(0)}k^{(d-1)})) \leq_T \gamma, \\
 & ((i^{(0)}k^{(d-1)}))((i^{(0)}j^{(0)})) \leq_T \gamma.
 \end{aligned}$$

With respect to the given ordering we have $((i^{(0)}j^{(0)})) < ((j^{(0)}k^{(d-1)})), ((j^{(0)}k^{(d-1)})) > ((i^{(0)}k^{(d-1)}))$ and $((i^{(0)}k^{(d-1)})) > ((i^{(0)}j^{(0)}))$.

Hence, the given ordering is a γ -compatible reflection ordering. \square

Example 4.3. Revisiting our running example, the γ -compatible reflection ordering for $G(3, 3, 3)$ as given in (21) is

$$\begin{aligned}
 (22) \quad & ((1^{(0)}2^{(0)})) < ((1^{(0)}3^{(0)})) < ((1^{(0)}3^{(2)})) < ((1^{(0)}3^{(1)})) < ((1^{(0)}2^{(2)})) \\
 & < ((2^{(0)}3^{(0)})) < ((2^{(0)}3^{(2)})) < ((2^{(0)}3^{(1)})).
 \end{aligned}$$

5. AUXILIARY RESULTS

This section contains some auxiliary results that help us proving Theorem 6.1. We first collect some results on the structure of $NC_{G(d,d,n)}$. Subsequently, we give some lemmas that explain how certain transformations of reduced T_γ -words of γ affect the descent set of the respective words.

5.1. The Structure of $NC_{G(d,d,n)}$. Unless otherwise stated, the following results were first observed by Athanasiadis, Brady and Watt [3] in the case of *real* reflection groups. Note that we write $NC_{G(d,d,n)}(\gamma)$ if we want to point out a specific choice of Coxeter element γ . Moreover, given a non-singleton interval $[u, v]$, we write $\lambda([u, v])$ for the set of label sequences of the maximal chains from u to v .

Lemma 5.1. *Let $[u, v]$ be a non-singleton interval in $NC_{G(d,d,n)}(\gamma)$ and denote by T_γ the set of all reflections in $NC_{G(d,d,n)}(\gamma)$.*

- (i) *If $[u, v]$ has length two and $(s, t) \in \lambda([u, v])$, then $(t, s') \in \lambda([u, v])$ for some $s' \in T_\gamma$.*
- (ii) *If $t \in T_\gamma$ appears in some coordinate of an element $\lambda([u, v])$, then $t = \lambda(u, u')$ for some covering relation (u, u') in $[u, v]$.*
- (iii) *The reflections appearing as the coordinates of an element of $\lambda([u, v])$ are pairwise distinct.*

Proof. (i) $(s, t) \in \lambda([u, v])$ implies that $v = ust = ut(t^{-1}st)$. It follows from [21, Proposition 2.9] that $t^{-1}st$ is a reflection and hence $\ell_T(t^{-1}st) = 1$. Since $[u, v]$ has length two, we know that $\ell_T(v) = \ell_T(u) + 2$. This implies that $t^{-1}st \leq_T v$ and hence $t^{-1}st \leq_T \gamma$.

Parts (ii) and (iii) follow from repeated application of part (i). \square

We omit the easy proof of the following fact.

Lemma 5.2. *Let $[u, v]$ be a non-singleton interval in $NC_{G(d,d,n)}(\gamma)$ and let $w = u^{-1}v$. The poset isomorphism $f : [\varepsilon, w] \rightarrow [u, v]$ given by $f(x) = ux$ satisfies $\lambda(x, y) = \lambda(f(x), f(y))$ for all covering relations (x, y) in $[\varepsilon, w]$.*

With the help of the previous results, it is possible to prove the following theorem.

Theorem 5.1. *Let γ be a Coxeter element of $G(d, d, n)$ and let λ be the natural edge labeling of $NC_{G(d,d,n)}(\gamma)$. For any total ordering of T_γ and any non-singleton interval $[u, v]$ in $NC_{G(d,d,n)}(\gamma)$ the lexicographically smallest maximal chain in $[u, v]$ is rising with respect to λ .*

Proof. We can carry over the proof of [3, Theorem 3.5(i)] essentially verbatim. \square

Since EL-shellability is a property that needs to be satisfied by every interval of a poset, it is helpful to understand the nature of the intervals of NC_W , for a well-generated complex reflection group W . Denote by V the complex vector space on which W acts. We call the maximal subgroup of W that fixes some $A \subseteq V$ pointwise *parabolic subgroup* of W . It follows from [29, Theorem 1.5] that the parabolic subgroup of W which fixes $A \subseteq V$, is generated by the reflections $t \in W$ that satisfy $A \subseteq \text{Fix}(t)$. Moreover, it follows from [5, Lemma 2.7] that a parabolic subgroup of W is again a well-generated complex reflection group. An analogous property holds for Coxeter elements.

Proposition 5.1 ([25, Proposition 6.3(i),(ii)]). *Let W be a well-generated complex reflection group and $w \in W$. Let T denote the set of all reflections of W . The following properties are equivalent:*

- (i) *w is a Coxeter element in a parabolic subgroup of W ;*
- (ii) *There is a Coxeter element γ_w of W such that $w \leq_T \gamma_w$.*

We call w parabolic Coxeter element if it satisfies one of these properties.

For some $w \in G(d, d, n)$, denote by $G(d, d, n)_w$ the parabolic subgroup of $G(d, d, n)$ in which w is a Coxeter element.

Lemma 5.3. *Let γ be a Coxeter element of $G(d, d, n)$. If $w \leq_T \gamma$, then any γ -compatible reflection ordering for $G(d, d, n)$ restricts to a w -compatible reflection ordering for $G(d, d, n)_w$.*

Proof. Let $w \leq_T \gamma$. We can assume that $\ell_T(w) \geq 2$. Let $t_1, t_2 \leq_T w$ be non-commuting reflections. The induced interval $I(t_1, t_2)$ is a subinterval of $[\varepsilon, w]$ and hence a subinterval of $[\varepsilon, \gamma]$. Let \prec be a γ -compatible reflection ordering for $G(d, d, n)$ and let $I(t_1, t_2)$ be of rank 2. By definition, there are two unique reflections $\tilde{t}_1, \tilde{t}_2 \in I(t_1, t_2) \cap T_\gamma$ such that $\tilde{t}_1 \tilde{t}_2 \leq_T \gamma$ implies that $\tilde{t}_1 \prec \tilde{t}_2$. Since $I(t_1, t_2)$ is a subinterval of $[\varepsilon, w]$, we know that $\tilde{t}_1, \tilde{t}_2 \leq_T w$ and since $I(t_1, t_2)$ has rank 2, the supremum of \tilde{t}_1 and \tilde{t}_2 is contained in $[\varepsilon, w]$ and equals either $\tilde{t}_1 \tilde{t}_2$ or $\tilde{t}_2 \tilde{t}_1$. Since t_1 and t_2 do not commute, we can conclude that \tilde{t}_1 and \tilde{t}_2 do not commute as well. Thus, it is clear that $\tilde{t}_1 \tilde{t}_2 \leq_T w$ and \prec restricts to a w -compatible reflection ordering for $G(d, d, n)_w$. \square

5.2. Shifting of Reduced Words. Since the reflections of $G(d, d, n)$ have order two, we can apply the results on shifted words that are generally valid for *real* reflection groups. Let us therefore recall the shifting lemma, as given in [1, Lemma 2.5.1].

Lemma 5.4 (THE SHIFTING LEMMA). *Let W be a complex reflection group, with the property that all reflections of W have order 2. Let (t_1, t_2, \dots, t_k) be a reduced T -word for $w \in W$, and let $1 < i < k$. Then the two sequences*

$$\begin{aligned} &(t_1, t_2, \dots, t_{i-2}, t_i, t_i t_{i-1} t_i, t_{i+1}, \dots, t_k) \\ &(t_1, t_2, \dots, t_{i-1}, t_i t_{i+1} t_i, t_i, t_{i+2}, \dots, t_k) \end{aligned}$$

are also reduced T -words for w . We will call these sequences left-shift respectively right-shift of (t_1, t_2, \dots, t_k) at (position) i .

Strictly speaking, Armstrong proved the shifting lemma for *real* reflection groups. Since all reflections of a *real* reflection group have order 2, we can carry over the proof of [1, Lemma 2.5.1] word by word.

It follows from the definition of λ that for any maximal chain c of $NC_{G(d, d, n)}(\gamma)$ the sequence of edge-labels $\lambda(c)$ is a reduced T_γ -word for γ . Unless otherwise stated, the Coxeter element γ which we consider in the remainder of this section is the one given in (19) and the descents (see Section 2.3) in the following lemmas refer to the ordering of T_γ as given in (21).

Lemma 5.5. *Let s_1, s_2, \dots, s_n be the simple reflections of $G(d, d, n)$ as given in (18). By definition, (s_1, s_2, \dots, s_n) is a reduced T_γ -word for γ . For every $1 < k \leq n$, the reduced T_γ -word $(s_1, \dots, s_{k-2}, s_k, s_k s_{k-1} s_k, s_{k+1}, \dots, s_n)$ has a descent at $k-1$.*

Proof. In the case $k < n$, we have

$$\begin{aligned} s_k s_{k-1} s_k &= ((k^{(0)}(k+1)^{(0)}))(((k-1)^{(0)}k^{(0)}))((k^{(0)}(k+1)^{(0)})) \\ &= (((k-1)^{(0)}(k+1)^{(0)})). \end{aligned}$$

According to (21), it follows that $((k-1)^{(0)}(k+1)^{(0)}) < ((k^{(0)}(k+1)^{(0)}))$, which is the desired descent. Now consider $k = n$:

$$\begin{aligned} s_n s_{n-1} s_n &= (((n-1)^{(0)}n^{(1)}))(((n-1)^{(0)}n^{(0)}))(((n-1)^{(0)}n^{(1)})) \\ &= (((n-1)^{(0)}n^{(2)})). \end{aligned}$$

Again, we have $((n-1)^{(0)}n^{(2)}) < (((n-1)^{(0)}n^{(1)}))$ in the ordering given in (21). \square

Lemma 5.6. *Let (t_1, t_2, \dots, t_n) be a reduced T_γ -word for γ that has a descent at k . The left-shift $(t_1, \dots, t_k, t_{k+2}, t_{k+2}t_{k+1}t_{k+2}, t_{k+3}, \dots, t_n)$ at $k+2$ has a descent at k or at $k+1$.*

Proof. Since there is a descent at k we know that $t_k > t_{k+1}$. If there is a descent at $k+1$, and hence $t_{k+1} > t_{k+2}$, the descent at k is preserved under the left-shift. Consider the case that $t_{k+1} < t_{k+2}$. Again, if $t_k > t_{k+2}$, we still have a descent at k . Thus it remains to show that there is a descent at $k+1$ if $t_k < t_{k+2}$. The proof proceeds by distinguishing several cases and showing that there is either a descent at $k+1$, or the situation cannot occur since $t_{k+1}t_{k+2} \not\leq \gamma$ and the respective word is no reduced T_γ -word for γ . We will provide the proof for the case $t_k = ((i^{(0)}j^{(0)}))$, where $1 \leq i < j < n$. The other cases can be shown in a similar fashion and are therefore left to the reader. By assumption, we have that $t_k > t_{k+1}$ and thus only two choices for t_{k+1} remain in the given case, namely (i) $((a^{(0)}b^{(0)}))$, where $1 \leq a < i < n$ and $a < b < n$ or (ii) $((i^{(0)}a^{(0)}))$, where $1 \leq i < a < j < n$. In both cases, we have three possibilities for t_{k+2} : (a) $((c^{(0)}n^{(s)}))$, for $1 \leq c \leq n$ and $0 \leq s < d$, (b) $((c^{(0)}e^{(0)}))$ for $1 \leq i < c < n$ or $1 < j < e < n$ and (c) $((c^{(0)}e^{(-1)}))$ for $1 \leq c < e < n$. We only need to examine the noncommuting combinations (otherwise the left-shift simply exchanges t_{k+1} and t_{k+2} and produces a descent at $k+1$ immediately)².

(i) Let $t_{k+1} = ((a^{(0)}b^{(0)}))$, where $1 \leq a < i < n$ and $a < b < n$. We obtain the following possibilities for t_{k+2} :

- | | | |
|--|--|------------------|
| (1) if $t_{k+2} = ((a^{(0)}n^{(s)}))$ | then $t_{k+2}t_{k+1}t_{k+2} = ((j^{(0)}n^{(s)}))$ | $> t_{k+2}$ |
| (2) if $t_{k+2} = ((b^{(0)}n^{(s)}))$ | then $t_{k+2}t_{k+1}t_{k+2} = ((a^{(0)}n^{(s)}))$ | $< t_{k+2}$ |
| (3) if $t_{k+2} = ((b^{(0)}e^{(0)}))$ | then $t_{k+2}t_{k+1}t_{k+2} = ((a^{(0)}e^{(0)}))$ | $< t_{k+2}$ |
| (4) if $t_{k+2} = ((a^{(0)}b^{(-1)}))$ | then $t_{k+2}t_{k+1}t_{k+2} = ((a^{(0)}b^{(-2)}))$ | $\nless t_{k+2}$ |
| (5) if $t_{k+2} = ((a^{(0)}e^{(-1)}))$ | then $t_{k+2}t_{k+1}t_{k+2} = ((b^{(0)}e^{(-1)}))$ | $> t_{k+2}$ |
| (6) | or $t_{k+2}t_{k+1}t_{k+2} = ((e^{(0)}b^{(1)}))$ | $\nless t_{k+2}$ |
| (7) if $t_{k+2} = ((b^{(0)}e^{(-1)}))$ | then $t_{k+2}t_{k+1}t_{k+2} = ((a^{(0)}e^{(-1)}))$ | $< t_{k+2}$ |
| (8) if $t_{k+2} = ((c^{(0)}a^{(-1)}))$ | then $t_{k+2}t_{k+1}t_{k+2} = ((c^{(0)}b^{(-1)}))$ | $> t_{k+2}$ |
| (9) if $t_{k+2} = ((c^{(0)}b^{(-1)}))$ | then $t_{k+2}t_{k+1}t_{k+2} = ((c^{(0)}a^{(-1)}))$ | $< t_{k+2}$ |

²Note that the colored transpositions are written in the sense that $((i^{(0)}j^{(s)}))$ is always meant to imply $i < j$.

The sign “ $\not\leq$ ” indicates that the cases in the fourth and sixth line can not occur. This is the case since the reflections $((a^{(0)}b^{(-2)}))$ and $((e^{(0)}b^{(1)}))$ are not contained in $NC_{G(d,d,n)}$. In lines 2, 3, 7 and 9, we obtain the desired descent at $k+1$. In the remaining cases, we can show analogously to the proof of Proposition 4.1 that $t_{k+1}t_{k+2} \not\leq_T \gamma$. Hence the corresponding cases cannot occur.

(ii) Let now $t_{k+1} = ((i^{(0)}a^{(0)}))$, where $1 \leq i < a < j < n$. Similarly to (i) we obtain the following possibilities for t_{k+2} :

- (1) if $t_{k+2} = ((i^{(0)}n^{(s)}))$ then $t_{k+2}t_{k+1}t_{k+2} = ((a^{(0)}n^{(s)})) > t_{k+2}$
- (2) if $t_{k+2} = ((a^{(0)}n^{(s)}))$ then $t_{k+2}t_{k+1}t_{k+2} = ((i^{(0)}n^{(s)})) < t_{k+2}$
- (3) if $t_{k+2} = ((a^{(0)}e^{(0)}))$ then $t_{k+2}t_{k+1}t_{k+2} = ((i^{(0)}e^{(0)})) < t_{k+2}$
- (4) if $t_{k+2} = ((i^{(0)}a^{(-1)}))$ then $t_{k+2}t_{k+1}t_{k+2} = ((i^{(0)}a^{(-2)})) \not\leq$
- (5) if $t_{k+2} = ((i^{(0)}e^{(-1)}))$ then $t_{k+2}t_{k+1}t_{k+2} = ((a^{(0)}e^{(-1)})) > t_{k+2}$
- (6) or $t_{k+2}t_{k+1}t_{k+2} = ((e^{(0)}a^{(1)})) \not\leq$
- (7) if $t_{k+2} = ((a^{(0)}e^{(-1)}))$ then $t_{k+2}t_{k+1}t_{k+2} = ((i^{(0)}e^{(-1)})) < t_{k+2}$
- (8) if $t_{k+2} = ((c^{(0)}i^{(-1)}))$ then $t_{k+2}t_{k+1}t_{k+2} = ((c^{(0)}a^{(-1)})) > t_{k+2}$
- (9) if $t_{k+2} = ((c^{(0)}a^{(-1)}))$ then $t_{k+2}t_{k+1}t_{k+2} = ((c^{(0)}i^{(-1)})) < t_{k+2}$

The argument works analogously to that of (i). \square

Lemma 5.7. *Let (t_1, t_2, \dots, t_n) be a reduced T_γ -word for γ that has a descent at k . The left-shift $(t_1, \dots, t_{k-1}, t_{k+1}, t_{k+1}t_k t_{k+1}, \dots, t_n)$ at $k+1$ has no descent at k if and only if*

- (23) $t_k = ((i^{(0)}j^{(0)}))$, $t_{k+1} = ((i^{(0)}a^{(0)}))$, where $1 \leq i < a < j < n$, or
- (24) $t_k = ((i^{(0)}j^{(-1)}))$, $t_{k+1} = ((i^{(0)}a^{(0)}))$, where $1 \leq i < a < j < n$, or
- (25) $t_k = ((i^{(0)}n^{(s)}))$, $t_{k+1} = ((i^{(0)}j^{(0)}))$.

Proof. In case (23) we obtain $t_{k+1}t_k t_{k+1} = ((a^{(0)}j^{(0)})) > t_{k+1}$. A computation as in the proof of Proposition 4.1 shows that $t_k t_{k+1} \leq_T \gamma$. Analogously, (24) yields $t_{k+1}t_k t_{k+1} = ((a^{(0)}j^{(-1)})) > t_{k+1}$ and $t_k t_{k+1} \leq_T \gamma$. Finally, (25) yields $t_{k+1}t_k t_{k+1} = ((j^{(0)}n^{(s)})) > t_{k+1}$. We can see that $t_k t_{k+1} \leq_T \gamma$.

Conversely, if we consider any other valid choice for t_k and t_{k+1} , analogously to the proof of Lemma 5.6 we can show that the descent at k is either preserved or the situation cannot occur in a reduced T_γ -word for γ . \square

Concluding this section, we show that the set of reduced T_γ -words for γ is connected under left-shifting. Note that a right-shift at k can be reversed by a left-shift at $k+1$ and that we can define left- and right-shifts for non-reduced T_γ -words for γ analogously to Lemma 5.4.

Lemma 5.8. *Let γ be a Coxeter element of $G(d, d, n)$ and let $w \leq_T \gamma$ with $\ell_T(w) = k$. Let (u_1, u_2, \dots, u_k) be a reduced T_w -word for w . A sequence (t_1, t_2, \dots, t_k) is a reduced T_w -word for w if and only if it can be obtained from (u_1, u_2, \dots, u_k) by a finite number of left-shifts.*

Proof. Let (u_1, u_2, \dots, u_k) be a reduced T_w -word for w . There is a maximal chain c in the interval $[\varepsilon, w]$ of $NC_{G(d,d,n)}(\gamma)$ such that $\lambda(c) = (u_1, u_2, \dots, u_k)$. Consider a

left-shift of $\lambda(c)$ at position l and define $x = u_1 u_2 \cdots u_{l-2}$ as well as $y = u_1 u_2 \cdots u_l$. Then, $[x, y]$ is an interval of length two and $(u_{l-1}, u_l) \in \lambda([x, y])$. By Lemma 5.1(i), we know that $(u_l, u_l u_{l-1} u_l) \in \lambda([x, y])$. Hence, the left-shift of $\lambda(c)$ yields the label sequence of some maximal chain in $[\varepsilon, w]$ and thus a reduced T_w -word for w . Repeating this procedure completes the "only if" part of the proof.

For the "if" part, consider a reduced T_w -word (t_1, t_2, \dots, t_k) for w and proceed by induction on k . For $k = 2$, the interval $[\varepsilon, w]$ is isomorphic to $NC_{G(e, e, 2)}$ for some $e \in \mathbb{N}$. We can conclude from (6) that $NC_{G(e, e, 2)}$ has $e + 2$ elements and thus e maximal chains. Let (u_1, u_2) be a reduced T_w -word for w . After e left-shifts at position 2, we obtain the reduced T_w -word $(u_2(u_1 u_2)^{e-1}, u_2(u_1 u_2)^e) = (u_1(u_1 u_2)^e, u_2(u_1 u_2)^e)$ for w . Since $G(e, e, 2)$ is isomorphic to the dihedral group of order $2e$ generated by u_1 and u_2 , we know that $(u_1 u_2)^e = \varepsilon$ and e is the smallest exponent with this property. Thus, each of the e left-shifts at position 2 yields a different T_w -word for w .

Hence we can assume that the statement is true for all $w \leq_T \gamma$ with $\ell_T(w) = k - 1$. Consider the factorization $wu_k = t_1 t_2 \cdots t_k u_k$. Let $u_k = ((i^{(0)} j^{(s)}))$, for parameters $i, j, s \in \mathbb{N}$ such that $u_k \leq_T w$. We first show that there must be a cycle of the form $((a^{(0)} \dots i^{(s_1)} \dots j^{(s_2)}))$ in $t_1 t_2 \cdots t_k$. For the sake of contradiction, assume that there is no such cycle. Then, $t_1 t_2 \cdots t_k$ must contain a cycle of one of the following forms: $((a^{(0)} \dots i^{(t)}))$, $[a^{(0)} \dots i^{(t)}]_b$, or $[a^{(0)} \dots i^{(t_1)} \dots j^{(t_2)}]_b$. Thus, $t_1 t_2 \cdots t_k u_k$ contains either a cycle of the form $((a^{(0)} \dots i^{(t)} j^{(s+t)}))$, a cycle of the form $[a^{(0)} \dots i^{(t)} j^{(s+t)}]_b$, or the product $[a^{(0)} \dots i^{(t_1)} \dots j^{(t_2)}]_b ((i^{(0)} j^{(s)}))$. In the first two cases, we have that $\dim \text{Fix}(t_1 t_2 \cdots t_k) > \dim \text{Fix}(t_1 t_2 \cdots t_k u_k)$, in the third case we have that $\dim \text{Fix}(t_1 t_2 \cdots t_k) = \dim \text{Fix}(t_1 t_2 \cdots t_k u_k)$. We can conclude from (7) that $\ell_T(t_1 t_2 \cdots t_k) = k \leq \ell_T(t_1 t_2 \cdots t_k u_k)$. On the other hand, since $u_k \leq_T w$ and $w = t_1 t_2 \cdots t_k$, we know that $\ell_T(t_1 t_2 \cdots t_k u_k) = k - 1$, which is a contradiction. Let $t_{i_1}, t_{i_2}, \dots, t_{i_l}$ denote the reflections among t_1, t_2, \dots, t_k that form a cycle of the form $((a^{(0)} \dots i^{(s_1)} \dots j^{(s_2)}))$ in $t_1 t_2 \cdots t_k$. By repeated right-shifts, we obtain a factorization $wu_k = \tilde{t}_1 \tilde{t}_2 \cdots \tilde{t}_{k-l} t_{i_1} t_{i_2} \cdots t_{i_l} u_k$. Clearly, $t_{i_1} t_{i_2} \cdots t_{i_l} u_k$ can be written as the product of $l - 1$ reflections v_1, v_2, \dots, v_{l-1} . Hence, $(\tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_{k-l}, v_1, v_2, \dots, v_{l-1})$ is a reduced T_{wu_k} -word for wu_k . By induction assumption, we can obtain this word from $(u_1, u_2, \dots, u_{k-1})$ by a finite number of left-shifts. Thus, we obtain $(\tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_{k-l}, v_1, v_2, \dots, v_{l-1}, u_k)$ from $(u_1, u_2, \dots, u_{k-1}, u_k)$ by a finite number of left-shifts as well. By construction of $(\tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_{k-l}, v_1, v_2, \dots, v_{l-1}, u_k)$ we know that (t_1, t_2, \dots, t_k) can be obtained from this word by a finite number of left-shifts. \square

Example 5.1. Figure 1 shows the left-shift graph of $NC_{G(3,3,3)}$. This is meant to be understood as follows: the top node represents the T_γ -word $s_1 s_2 s_3$ and two T_γ -words p and q are connected by an arrow from p to q if q can be obtained from p by a left-shift. For better readability, we have omitted directed cycles.

6. EL-SHELLABILITY OF $NC_{G(d,d,n)}$

We have proved in the previous section that left-shifting a given reduced T_γ -word for γ reduces the number of descents only in a few cases. We use this fact to prove the EL-shellability of $NC_{G(d,d,n)}$.

Theorem 6.1. *Let γ be the Coxeter element of $G(d, d, n)$ as defined in (19) and let T_γ be the set of all reflections $t \in G(d, d, n)$ that satisfy $t \leq_T \gamma$. Let furthermore*

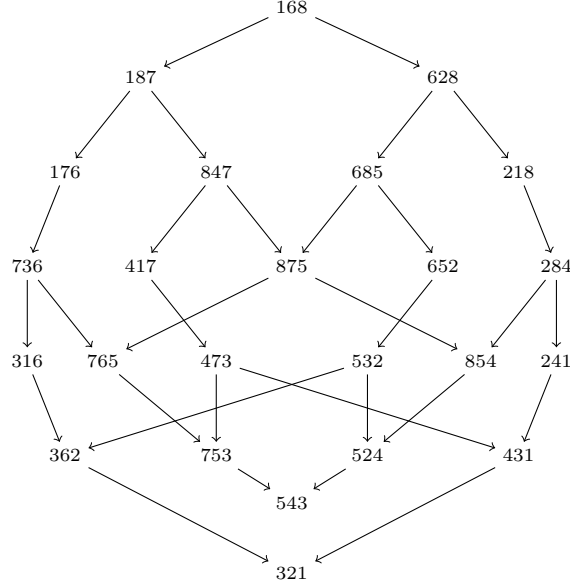


FIGURE 1. The left-shift graph of $NC_{G(3,3,3)}$. The nodes represent reduced T_γ -words for γ , where every reflection is replaced by its position in the ordering (22).

$\lambda : \mathcal{E}(NC_{G(d,d,n)}) \rightarrow T_\gamma$ be the natural labeling function of $NC_{G(d,d,n)}$ that maps an edge (u, v) to the reflection $u^{-1}v$. If T_γ is ordered as in (21), λ is an EL-labeling for $NC_{G(d,d,n)}$.

Proof. According to Theorem 5.1, the lexicographically smallest chain in every interval of $NC_{G(d,d,n)}$ is rising for any ordering of T_γ . Thus, it only remains to show that there is at most one rising chain in every interval. By Lemma 5.2, it is sufficient to consider intervals of the form $[\varepsilon, w]$. Proposition 5.1 states that w is a Coxeter element in the parabolic subgroup $G(d, d, n)_w$. Theorem 1.5 in [29] implies that $G(d, d, n)_w$ is generated by a subset of the reflections of $G(d, d, n)$. By Lemma 5.3 we know that the restriction of the ordering in (21) to $G(d, d, n)_w$ yields a w -compatible reflection ordering. Hence, it is sufficient to consider the interval $[\varepsilon, \gamma]$. Let s_1, s_2, \dots, s_n be the simple reflections of $G(d, d, n)$ as given in (18). The reduced T_γ -word (s_1, s_2, \dots, s_n) is rising with respect to the ordering given in (21). At the same time we notice that any other permutation of the simple reflections cannot yield a rising labeling. Any other permutation of s_1, s_2, \dots, s_n does not even yield a reduced T_γ -word for γ . So the remaining task is to show that a maximal chain c whose label $\lambda(c) = (t_1, t_2, \dots, t_n)$ is no permutation of simple reflections cannot be rising.

It follows from Lemma 5.8 that every reduced T_γ -word of γ can be obtained from (s_1, s_2, \dots, s_n) by a finite number of left-shifts. If the reduced T_γ -word $(t_1, \dots, t_{k-1}, t_k, t_{k+1}, \dots, t_n)$ has a descent at position k , then the corresponding left-shift at k has an inversion at $k-1$ and hence a descent at $k-1$ or k . Lemma 5.6 shows that a left-shift at $k+2$ does not reduce the number of descents. In view of

Lemma 5.7, we notice that there are only three cases in which a left-shift at $k + 1$ removes the descent at k .

(i) $t_k = ((i^{(0)}j^{(0)}))$, $t_{k+1} = ((i^{(0)}a^{(0)}))$, where $1 \leq i < a < j < n$. Let \tilde{a} be the colored integer that is sent to $a^{(0)}$ by $t_{k+2}t_{k+3} \cdots t_n$. Clearly, $t_k t_{k+1} \cdots t_n$ sends \tilde{a} to $j^{(0)}$. If $\tilde{a} = n^{(s)}$, there must be reflections among t_1, \dots, t_{k-1} forming the cycle $((j^{(0)} \dots n^{(t)}))$. One of these reflections must be larger than t_{k+1} . Now consider $\tilde{a} = (j-1)^{(0)}$. Hence, there must be reflections forming the cycle $((a^{(0)} \dots (j-1)^{(s)}))$ among t_{k+2}, \dots, t_n . At least one of these reflections is smaller than $t_{k+1}t_k t_{k+1} = ((a^{(0)}j^{(0)}))$. Only the case $\tilde{a} = b^{(s)}$ remains, where $1 \leq b < n$ is not considered above. Hence, there must be a cycle $((j^{(0)} \dots (b+1)^{(s)}))$, formed by some reflections among t_1, \dots, t_{k-1} . At least one of the reflections forming this cycle must be larger than t_k . So in each case there is (at least) one inversion in the left-shift.

(ii) $t_k = ((i^{(0)}j^{(-1)}))$, $t_{k+1} = ((i^{(0)}a^{(0)}))$, where $1 \leq i < a < j < n$. This case works analogously to (i).

(iii) $t_k = ((i^{(0)}n^{(s)}))$, $t_{k+1} = ((i^{(0)}j^{(0)}))$. Let \tilde{a} be the colored integer that is sent to $n^{(t)}$ by $t_{k+2}t_{k+3} \cdots t_n$. Analogously to (i), we notice that there must be at least one inversion in the respective left-shift.

The previous paragraphs show that any left-shift of a reduced T_γ -word for γ that already contains a descent, has at least one inversion and thus at least one descent. Finally, Lemma 5.5 concludes the proof by implying that any left-shift of (s_1, s_2, \dots, s_n) creates a descent. \square

Example 6.1. Figure 2 shows the lattice $NC_{G(3,3,3)}$. The given integer labeling is derived from the natural labeling λ by mapping every reflection to its position in the reflection ordering given in (22). We notice that this is an EL-labeling, where the unique rising chain in the interval $[\varepsilon, \gamma]$ is indicated with thick lines.

7. EL-SHELLABILITY OF NC_W FOR THE EXCEPTIONAL GROUPS W

In this section, we provide explicit orderings of the reflections of the exceptional well-generated complex reflection groups such that the natural labeling function is an EL-labeling of the respective noncrossing partition lattice. It turns out that the noncrossing partition lattice of most of these groups is isomorphic to the noncrossing partition lattice of some *real* reflection group. Only five groups, namely $G_{24}, G_{27}, G_{29}, G_{33}$ and G_{34} , remain unrelated to any known case. For these cases we have proved EL-shellability by means of a computer program (LINS) that can be obtained from <http://homepage.univie.ac.at/henri.muehle/misc.php>. LINS utilizes the property that for every total ordering of the reflections of an exceptional well-generated complex reflection group the lexicographically smallest chain in every interval is rising. It takes an arbitrary ordering of the reflections and checks which intervals have more than one rising chain and adapts the ordering such that the additional rising chains vanish. However, this algorithm is not deterministic, meaning that different runs of LINS produce different orderings. It uses Jean Michel's GAP-distribution³ for setting up the reflection groups and Daniel Borchmann's FCA-tool⁴ for computing the chains of the lattice. For more information on Formal Concept Analysis (FCA), we refer to the standard monograph [17] by Bernhard Ganter and Rudolf Wille. LINS outputs several files, including some

³To be found at <http://www.math.jussieu.fr/~jmichel/gap3/>.

⁴To be found at <http://www.math.tu-dresden.de/~borch//conexp-clj/>.

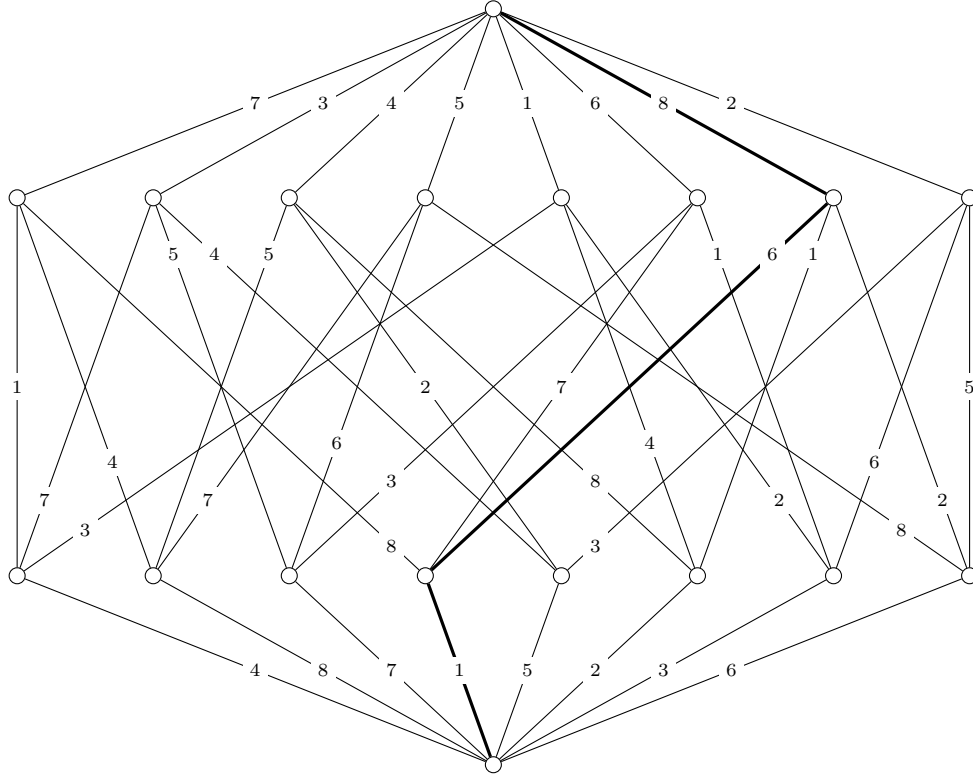


FIGURE 2. The lattice of noncrossing partitions of $G(3,3,3)$ and the respective EL-labeling. The integer labels correspond to the position of the reflections in (22).

GAP scripts, a file containing the labeled chains, as well as a file containing the final ordering of the reflections. The reflections are abstractly named in the form s_k , where k is an integer between 1 and $|NC_W|$. The value k that is assigned to a certain reflection depends on the position at which GAP lists this reflection in its internal representation of the group elements. This naming of the reflections is deterministic, so that we can identify the actual group elements behind the names with GAP and the respective GAP script⁵.

The main result of this section is proved in the subsequent paragraphs explicitly.

Theorem 7.1. *Let W be an exceptional well-generated complex reflection group. Then NC_W is EL-shellable.*

The Groups $G_{23}, G_{28}, G_{30}, G_{35}, G_{36}, G_{37}$. These groups are the six exceptional *real* reflection groups [14, p. 6]. Hence their noncrossing partition lattices are EL-shellable by [3, Theorem 1.1].

⁵There is a file named `lins` included in the zip-archive containing LINS. Moreover, this script can be downloaded separately from <http://homepage.univie.ac.at/henri.muehle/files/lins>.

The Groups G_{25}, G_{26}, G_{32} . We can convince ourselves that the following isomorphisms hold:

$$\begin{aligned} NC_{G_{25}} &\cong NC_{G(1,1,3)}, \\ NC_{G_{26}} &\cong NC_{G(2,1,3)}, \\ NC_{G_{32}} &\cong NC_{G(1,1,4)}. \end{aligned}$$

Since $G(1, 1, 3)$, $G(2, 1, 3)$ and $G(1, 1, 4)$ are *real* reflection groups, the EL-shellability of the respective noncrossing partition lattices follows from [3, Theorem 1.1].

Remark 7.1. In order to prove that two lattices \mathbb{P}, \mathbb{L} are isomorphic, it is sufficient to give two isomorphisms α, β

$$(26) \quad \alpha : J(\mathbb{P}) \rightarrow J(\mathbb{L}) \quad \text{and} \quad \beta : M(\mathbb{P}) \rightarrow M(\mathbb{L})$$

that satisfy the property that $j \leq_{\mathbb{P}} m$ if and only if $\alpha(j) \leq_{\mathbb{L}} \beta(m)$ for all $j \in J(\mathbb{P})$ and $m \in M(\mathbb{P})$. The sets J and M denote the join- respectively meet-irreducible elements of the corresponding lattice.

Clearly, $J(NC_W)$ is the set of atoms and $M(NC_W)$ is the set of co-atoms of NC_W for any well-generated complex reflection group W .

The Groups $G_4, G_5, G_6, G_8, G_9, G_{10}, G_{14}, G_{16}, G_{17}, G_{18}, G_{20}, G_{21}$. All of these groups are of rank 2. Hence, the respective lattice of noncrossing partitions has rank 2 and is thus isomorphic to $NC_{G(k,k,2)}$, where k is the number of reflections below a Coxeter element in the respective group. Groups of the form $G(k, k, 2)$ are *real* reflection groups and hence the desired property follows from [3, Theorem 1.1].

The Groups $G_{24}, G_{27}, G_{29}, G_{33}, G_{34}$. As described in the beginning of this section, we provide an explicit ordering for these groups that was computed with LINS. The abstract encodings listed in Table 1 can be resolved with the GAP script provided by LINS. Note that the given orderings are just one possibility to make the natural labeling an EL-labeling.

8. EL-SHELLABILITY OF m -DIVISIBLE NONCROSSING PARTITIONS

Up to now, we have shown that the lattices of noncrossing partitions are EL-shellable for all well-generated complex reflection groups. Bearing this result in mind, we are able to finally prove Theorem 1.1.

Proof of Theorem 1.1. It follows from Theorem 6.1 and Theorem 7.1 as well as [3, Theorem 1.1] that NC_W is EL-shellable, for every well-generated complex reflection group W . Hence, we can construct an EL-labeling for $NC_W^{(m)} \cup \{\hat{0}\}$ in the same way as described in [1, Theorem 3.7.2]. \square

9. APPLICATIONS

EL-shellability of a partially ordered set implies a certain structure of the associated order complex. In the present case, this structure was already conjectured in [2] and can now be proved. Recall that the Fuß-Catalan numbers $\text{Cat}^{(m)}(W)$, see (6), count the m -divisible noncrossing partitions associated to a well-generated complex reflection group W for some $m \in \mathbb{N}$.

Group	Ordering of the reflections
G_{24}	$s_{26} < s_5 < s_3 < s_{29} < s_{21} < s_{28} < s_{18} < s_7 < s_2 < s_4 < s_{11} < s_8$ $< s_{23} < s_{25}$
G_{27}	$s_{23} < s_{38} < s_{42} < s_{15} < s_{36} < s_{29} < s_{33} < s_{27} < s_{18} < s_{13} < s_4$ $< s_3 < s_2 < s_8 < s_5 < s_{21} < s_{17} < s_{34} < s_{37} < s_{30}$
G_{29}	$s_{101} < s_4 < s_{76} < s_{109} < s_8 < s_{105} < s_{64} < s_{47} < s_6 < s_{33} < s_{68}$ $< s_{13} < s_{20} < s_{39} < s_{93} < s_9 < s_{88} < s_2 < s_{70} < s_{28} < s_{110}$ $< s_{25} < s_{53} < s_3 < s_{18}$
G_{33}	$s_5 < s_{13} < s_7 < s_{33} < s_{56} < s_{19} < s_{36} < s_{58} < s_{47} < s_{182} < s_{16}$ $< s_{17} < s_{224} < s_{281} < s_{297} < s_{42} < s_{179} < s_{217} < s_{89} < s_{128}$ $< s_{86} < s_{110} < s_2 < s_{172} < s_{277} < s_{169} < s_{76} < s_{68} < s_3 < s_{12}$
G_{34}	$s_{1568} < s_{937} < s_{1361} < s_{213} < s_{13} < s_{142} < s_{669} < s_{888} < s_{58} < s_7$ $< s_{65} < s_{67} < s_{480} < s_{295} < s_8 < s_{37} < s_{40} < s_{256} < s_{714}$ $< s_{1060} < s_{1447} < s_{17} < s_3 < s_{117} < s_{53} < s_{1252} < s_{639} < s_{62}$ $< s_6 < s_{702} < s_{915} < s_{1043} < s_{43} < s_{359} < s_{428} < s_{23} < s_4$ $< s_{75} < s_{127} < s_{191} < s_{368} < s_{157} < s_{648} < s_{1234} < s_{181} < s_2$ $< s_{683} < s_{49} < s_{264} < s_{235} < s_{905} < s_{1241} < s_{60} < s_{1558} < s_{1353}$ $< s_{319}$

TABLE 1. Explicit orderings of the reflections for the remaining groups that make the natural labeling λ an EL-labeling.

Corollary 9.1. *Let W be a well-generated complex reflection group of rank n and let m be a positive integer. The order complex of the poset $NC_W^{(m)}$ with maximal and minimal elements removed is homotopy equivalent to a wedge of $\left(Cat^{(-m-1)}(W) - Cat^{(-m)}(W)\right)$ -many $(n-2)$ -spheres.*

Proof. Removing maximal and minimal elements from $NC_W^{(m)}$ yields a rank-selected subposet of $NC_W^{(m)}$. Theorem 1.1 and [8, Theorem 4.1] imply that this truncated poset is shellable. Hence, the order complex associated to $NC_W^{(m)}$ with maximal and minimal elements removed is also shellable. Theorem 9 in [2] then implies the result. \square

The previous result has consequences for the Möbius function of $NC_W^{(m)}$.

Corollary 9.2. *Let W be a well-generated complex reflection group of rank n and let γ be a Coxeter element of W . Denote by M the set of minimal elements of $NC_W^{(m)}(\gamma)$. Consider the lattice $\left(NC_W^{(m)}(\gamma) \setminus M\right) \cup \{\hat{0}\}$ that arises from $NC_W^{(m)}(\gamma) \setminus$*

M by adding a unique minimal element $\hat{0}$. For all positive integers m , we have $\mu(\hat{0}, \gamma) = (-1)^n \left(\text{Cat}^{(-m-1)}(W) - \text{Cat}^{(-m)}(W) \right)$.

Proof. Theorem 1.1 implies that there exists an EL-labeling for $(NC_W^{(m)}(\gamma) \setminus M) \cup \{\hat{0}\}$ for any well-generated complex reflection group W . Hence, the proof of this corollary works analogously to the proof of [30, Theorem 1.1]. \square

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